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ON THE TORSIONAL IMPACT OF A THICK ELASTIC PLATE

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Abstract—Propagation of the torsional waves in a thick elastic plate is analyzed under torsional impact loading on the circular regions of the upper and the lower surfaces. The solution is obtained by means of the Laplace and the Hankel transforms. Each stress component is represented by the sum of terms involving elliptic integrals. The interference and the reflection of the torsional waves in the plate are considered in detail. Numerical results for the variations of the stresses with time are shown graphically.

1. INTRODUCTION

It is very important in practice to investigate the dynamic problem of torsional impact. On account of this, some investigators have studied problems of this kind. Reissner [1] first dealt with the torsional vibration problem on an elastic semi-infinite solid subjected to a surface torque, initially undisturbed, which varies periodically with time and acts on a circular region of the surface of the solid. Reissner and Sagoci [2] analyzed the problem of a semi-infinite solid loaded by a periodic surface displacement on a circular region and free from loading on the remaining part of the boundary. Eason [3] has obtained a general solution of a semi-infinite solid for torsional impact loading and has considered in detail for three particular types of impulsive loading. He also discussed briefly with the step-function loading.

In the present paper, the author treats the torsional impact loading of a thick elastic plate, which is initially undisturbed. The solution is obtained by means of the Laplace and the Hankel transforms. The inverse Laplace transform is obtained by using the convolution theorem. Stress components are represented by the sum of terms involving elliptic integrals. The interference and the reflection of the stress waves in the plate are considered in detail.

2. ANALYSIS

Let (r, θ, z) be the cylindrical coordinates as shown in Fig. 1 and let the thickness of the plate be $2h$. We assume that the step-function type of surface shear loads $(\tau_{\theta z})_{z=\pm h} = \tau_0 r H(a-r)H(t)/a$ act on the circular regions of radius a on both surfaces of the plate and the remaining ones are free from loading. From the axisymmetry of the torsional impact to the plate, all quantities depend on r, z and the time t. The non-zero displacement is the circumferential v, and the stresses corresponding to it are the shear stresses $\tau_{\theta z}$ and $\tau_{r\theta}$. Then the equation of motion is

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}+\frac{\partial^2 v}{\partial z^2}=\frac{1}{c_2^2}\frac{\partial^2 v}{\partial t^2} \tag{1}$$

where $c_2[=(gG/\gamma)^{1/2}]$ is the velocity of the shear wave and G, γ, g are respectively the shear modulus, the weight per unit volume and the acceleration of gravity.

Introducing a stress function $\lambda_3(r, z, t)$ defined as

$$v=-\frac{\partial \lambda_3}{\partial r} \tag{2}$$

and substituting equation (2) into equation (1), we can get λ_3 as the solution of the wave equation

$$\nabla^2\lambda_3=\frac{1}{c_2^2}\frac{\partial^2\lambda_3}{\partial t^2} \tag{3}$$

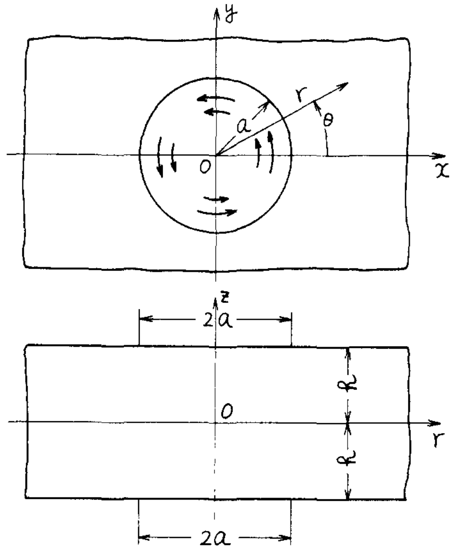

Fig. 1. The thick plate subjected to torsional impact.

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

$\tau_{\theta z}$ and $\tau_{r\theta}$ are given by using equation (2),

$$\left.\begin{aligned} \frac{\tau_{\theta z}}{G} &= -\frac{\partial^2 \lambda_3}{\partial r \partial z}, \\ \frac{\tau_{r\theta}}{G} &= \frac{2}{r}\frac{\partial \lambda_3}{\partial r} + \frac{\partial^2 \lambda_3}{\partial z^2} - \nabla^2 \lambda_3. \end{aligned}\right\} \tag{4}$$

The initial conditions are written as follows:

$$(\lambda_3)_{t=0} = (\partial \lambda_3/\partial t)_{t=0} = 0 \tag{5}$$

and the Laplace transform of equation (3) with respect to t is

$$\nabla^2 \bar{\lambda}_3 = \alpha^2 \bar{\lambda}_3 \tag{6}$$

where

$$\bar{\lambda}_3(r, z, p) = \int_0^\infty \lambda_3 e^{-pt}\, dt, \quad \alpha = \frac{p}{c_2}.$$

Considering that $\tau_{\theta z}$ is an even function with respect to z, we can get as the solution of Eq. (6) for the problem

$$2G\bar{\lambda}_3 = \int_0^\infty A(\xi) J_0(\xi r) \sinh \beta z\, d\xi, \quad \beta = \sqrt{(\xi^2 + \alpha^2)} \tag{7}$$

where $J_n(\xi r)$ is the Bessel function of the first kind, and $A(\xi)$ is an arbitrary function determined by the boundary condition.

From the Laplace transform of the boundary condition, we obtain

$$\frac{1}{2}\int_0^\infty \beta \xi A(\xi) J_1(\xi r) \cosh \beta h\, d\xi = \tau_0 \frac{r}{a}\frac{1}{p} H(a - r) \tag{8}$$

where $H(x)$ is the Heaviside's step function. After $A(\xi)$ is determined by means of the inverse Hankel transform of equation (8), the Laplace transforms $\bar{v}$, $\bar{\tau}_{\theta z}$, $\bar{\tau}_{r\theta}$ of v, $\tau_{\theta z}$, $\tau_{r\theta}$ are given respectively as follows:

$$\left.\begin{aligned}\bar{v}&=\frac{\tau_0 a}{G}\int_0^\infty \frac{\sin\beta z}{p\beta\cosh\beta h}J_1(\xi r)J_2(\xi a)\,d\xi,\\ \bar{\tau}_{\theta z}&=\tau_0 a\int_0^\infty \frac{\cosh\beta z}{p\cosh\beta h}J_1(\xi r)J_2(\xi a)\,d\xi,\\ \bar{\tau}_{r\theta}&=-\tau_0 a\int_0^\infty \frac{\sinh\beta z}{p\beta\cosh\beta h}\xi J_2(\xi r)J_2(\xi a)\,d\xi.\end{aligned}\right\}\tag{9}$$

3. INVERSE LAPLACE TRANSFORMATION

Now, we expand $(\cosh\beta h)^{-1}$ to an infinite series of $\exp(-2\beta h)$ and obtain

$$\left.\begin{aligned}\frac{\sinh\beta z}{p\beta\cosh\beta h}&=\frac{1}{p\beta}\sum_{n=0}^\infty(-1)^n(e^{-\beta x_n}-e^{-\beta y_n}),\\ \frac{\cosh\beta z}{p\cosh\beta h}&=\frac{1}{p}\sum_{n=0}^\infty(-1)^n(e^{-\beta x_n}+e^{-\beta y_n})\end{aligned}\right\}\tag{10}$$

where

$$x_n=(h-z)+2nh,(\geqq 0).\quad y_n=(h+z)+2nh,(\geqq 0).$$

Here, if we use the inverse Laplace transform formulae

$$\left.\begin{aligned}L^{-1}\left\{\frac{1}{\beta}e^{-\beta x_n}\right\}&=c_2J_0(\xi\sqrt{(\tau^2-x_n^2)})H(\tau-x_n),\\ L^{-1}(e^{-\beta x_n}\}&=\delta(\tau-x_n)-\frac{c_2\xi x_n}{\sqrt{(\tau^2-x_n^2)}}J_1(\xi\sqrt{(\tau^2-x_n^2)})H(\tau-x_n)\end{aligned}\right\}\tag{11}$$

where $\tau=c_2t$, and $\delta(x)$ is the Dirac delta function, the inverse Laplace transforms of $\bar{\tau}_{\theta z}$ and $\bar{\tau}_{r\theta}$ in equation (9) will be obtained by the convolution theorem as follows:

$$\left.\begin{aligned}\frac{\tau_{\theta z}}{\tau_0 a}&=\sum_{n=0}^\infty(-1)^n\left[\frac{r}{a^2}H(a-r)\{H(\tau-x_n)+H(\tau-y_n)\}-\{g(x_n)+g(y_n)\}\right],\\ \frac{\tau_{r\theta}}{\tau_0 a}&=-\sum_{n=0}^\infty(-1)^n\{f(x_n)-f(y_n)\}\end{aligned}\right\}\tag{12}$$

where

$$\left.\begin{aligned}f(x_n)&=H(\tau-x_n)\int_{x_n}^\tau d\rho\int_0^\infty \xi J_0(\xi\sqrt{(\rho^2-x_n^2)})J_2(\xi r)J_2(\xi a)\,d\xi,\\ g(x_n)&=H(\tau-x_n)\int_{x_n}^\tau\frac{x_n}{\sqrt{(\rho^2-x_n^2)}}d\rho\int_0^\infty\left\{\frac{2}{a}J_1(\xi a)-\xi J_0(\xi a)\right\}\\ &\quad\times J_1(\xi\sqrt{(\rho^2-x_n^2)})J_1(\xi r)\,d\xi.\end{aligned}\right\}\tag{13}$$

To integrate equation (13) with respect to ξ, we use the integral formula[4]

$$\int_0^\infty \xi^{1-\mu}J_\mu(a\xi)J_\nu(b\xi)J_\nu(c\xi)\,d\xi=\begin{cases}0,(a<|b-c|\text{ and }a>b+c),\\ \dfrac{(bc)^{\mu-1}}{\sqrt{(2\pi)}a^\mu}\sin^{\mu-1/2}\varphi\,e^{(\mu-1/2)\pi i}p_{\nu-1/2}^{1/2-\mu}(\cos\varphi),\\ (|b-c|<a<b+c)\end{cases}\tag{14}$$

where

$$a, b, c > 0.\ \mu > -\tfrac{1}{2},\ \nu > -1,\ |\mu - \nu|: \text{an integer},$$
$$\varphi = \cos^{-1}\{(b^2 + c^2 - a^2)/2bc\},$$

and $P^{1/2-\mu}_{\nu-1/2}(\cos\varphi)$ is the associated Legendre function. Thus, $f(x_n)$ and $g(x_n)$ are non-zero only in $|a-r| < \sqrt{(\rho^2 - x_n^2)} < a + r$ and are given in this interval as follows:

$$\left.\begin{aligned} f(x_n) &= \frac{1}{\pi a r} H(\tau - x_n) \int_{x_n}^{\tau} \frac{\cos 2\varphi'}{\sin\varphi'} \mathrm{d}\rho', \\ g(x_n) &= \frac{x_n}{\pi} H(\tau - x_n) \int_{x_n}^{\tau} \frac{1}{\sqrt{(\rho^2 - x_n^2)}} \left\{ \frac{2}{a^2} \sin\varphi - \frac{\cot\varphi}{r\sqrt{(\rho^2 - x_n^2)}} \right\} d\rho \end{aligned}\right\} \tag{15}$$

where

$$\varphi = \cos^{-1} \frac{r^2 - a^2 + \rho^2 - x_n^2}{2r\sqrt{(\rho^2 - x_n^2)}}, \quad \phi' = \cos^{-1} \frac{a^2 + r^2 + x_n^2 - \rho^2}{2ar}.$$

Putting

$$a_n = \sqrt{((a-r)^2 + x_n^2)}, \quad b_n = \sqrt{((a+r)^2 + x_n^2)},$$
$$a_n' = \sqrt{((a-r)^2 + y_n^2)}, \quad b_n' = \sqrt{((a+r)^2 + y_n^2)},$$

we get

$$\sin\varphi = \frac{\sqrt{(b_n^2 - \rho^2)(\rho^2 - a_n^2)}}{2r\sqrt{(\rho^2 - x_n^2)}}, \quad \sin\varphi' = \frac{\sqrt{(b_n^2 - \rho^2)(\rho^2 - a_n^2)}}{2ar}.$$

Then, we can rewrite equation (15) as follows:

$$\left.\begin{aligned} f(x_n) &= \frac{1}{\pi a^2 r^2} H(\tau - x_n) H(\tau - a_n) \int_{a_n}^{\min(\tau, b_n)} \frac{2a^2 r^2 - (b_n^2 - \rho^2)(\rho^2 - a_n^2)}{\sqrt{((b_n^2 - \rho^2)(\rho^2 - a_n^2))}} \mathrm{d}\rho, \\ g(x_n) &= \frac{x_n}{\pi a^2 r} H(\tau - x_n) H(\tau - a_n) \\ &\quad \times \int_{a_n}^{\min(\tau, b_n)} \frac{(b_n^2 - \rho^2)(\rho^2 - a_n^2) - a^2(\rho^2 - x_n^2) + a^2(a^2 - r^2)}{(\rho^2 - x_n^2)\sqrt{(b_n^2 - \rho^2)(\rho^2 - a_n^2)}} \mathrm{d}\rho. \end{aligned}\right\} \tag{16}$$

Moreover, to express the integrals with respect to ρ of equation (16) by elliptic integrals, we put

$$k^2 = 1 - \frac{a_n^2}{b_n^2}, \quad \rho^2 = b_n^2(1 - k^2 \sin^2\zeta),$$
$$b_n^2 - \rho^2 = b_n^2 k^2 \sin^2\zeta, \quad \rho^2 - a_n^2 = b_n^2 k^2 \cos^2\zeta,$$
$$\rho^2 - x_n^2 = (b_n^2 - x_n^2)(1 - A_0 \sin^2\zeta), \quad A_0 = 4ar/(a+r)^2,$$

and

$$\zeta = \frac{\pi}{2}, \quad \text{at } \rho = a_n,$$
$$\zeta = \zeta_0 = \sin^{-1}\{\sqrt{(1 - \min(1, \tau^2/b_n^2))}/k\}, \quad \text{at } \rho = \min(\tau, b_n).$$

Then, equation (16) becomes

$$\left.\begin{aligned} f(x_n) &= \frac{1}{3\pi a^2 r^2 b_n} H(\tau - x_n) H(\tau - a_n) \\ &\quad \times \{2(3a^2 r^2 + a_n^2 b_n^2) I_1 - b_n^2(a_n^2 - b_n^2) I_2 - \frac{1}{2} b_n^2 (b_n^2 - a_n^2) \sqrt{(1 - k^2 \sin^2 z_0)} \sin 2\zeta_0), \\ g(x_n) &= \frac{x_n}{\pi a^2 r b_n} H(\tau - x_n) H(\tau - a_n) \\ &\quad \times \left\{(a_n^2 + b_n^2 - x_n^2 - a^2) I_1 - b_n^2 I_2 + \frac{r^2(a-r)}{a+r} I_3\right\}, \end{aligned}\right\} \quad (17)$$

where

$$I_1 = \int_{\zeta_0}^{\pi/2} \frac{d\zeta}{\sqrt{(1 - k^2 \sin^2 \zeta)}} = F\left(\frac{\pi}{2}, k\right) - F(\zeta_0, k),$$

$$I_2 = \int_{\zeta_0}^{\pi/2} \sqrt{(1 - k^2 \sin^2 \zeta)}\, d\zeta = E\left(\frac{\pi}{2}, k\right) - E(\zeta_0, k),$$

$$I_3 = \int_{\zeta_0}^{\pi/2} \frac{d\zeta}{(1 - A_0 \sin^2 \zeta)\sqrt{(1 - k^2 \sin^2 \zeta)}} = \Pi\left(\frac{\pi}{2}; -A_0, k\right) - \Pi(\zeta_0; -A_0, k)$$

and $F(\zeta, k)$, $E(\zeta, k)$, $\Pi(\zeta; -A_0, k)$ are respectively the elliptic integrals on the first, second and third kinds. Similarly, $f(y_n)$ and $g(y_n)$ are obtained by replacing a'_n, b'_n and y_n for a_n, b_n and x_n in equation (17) respectively. The stresses are easily calculated by equations (12) and (17).

In particular, the stresses on the middle plane of the plate are

$$\tau_{r\theta} = 0, \quad \frac{\tau_{\theta z}}{\tau_0 a} = 2 \sum_{n=0}^{\infty} (-1)^n \left\{\frac{r}{a^2} H(a - r) H(\tau - x_n) - g(x_n)\right\}. \quad (18)$$

The stresses on the surface are, for example,

$$\left.\begin{aligned} \frac{(\tau_{\theta z})_{z=h}}{\tau_0 a} &= \frac{r}{a^2} H(a - r) H(\tau), \\ \frac{(\tau_{r\theta})_{z=h}}{\tau_0 a} &= -f(x_0) - 2 \sum_{n=1}^{\infty} (-1)^n f(x_n). \end{aligned}\right\} \quad (19)$$

At $r = a$ on the surface, $(\tau_{r\theta})_{z=h,r=a}$ tends to negative infinity, because

$$\{f(x_0)\}_{z=h,r=a} = \frac{1}{\pi a} H(\tau) \left[\ln \left|\frac{\rho}{2a + \sqrt{(4a^2 - \rho^2)}}\right| - \frac{\sqrt{(4a^2 - \rho^2)^3}}{3a^2}\right]_{\rho=0}^{\rho=\min(\tau, 2a)}.$$

4. STRESS WAVE FIELDS AND NUMERICAL RESULTS

The stress wave fields can be determined by regions in which the stresses are non-zero. The non-zero regions of each term in equation (12) are

$$\left.\begin{aligned} &\frac{r}{a^2} H(a - r) H(\tau - x_n) \rightarrow \tau \geqq 2nh,\ h \geqq z \geqq -h,\ \tau \geqq x_n,\ a \geqq r, \\ &\frac{r}{a^2} H(a - r) H(\tau - y_n) \rightarrow \tau \geqq 2nh,\ h \geqq z \geqq -h,\ \tau \geqq x_n,\ a \geqq r. \\ &\quad g(x_n) \text{ or } f(x_n) \rightarrow \tau \geqq 2nh,\ h \geqq z \geqq -h,\ \tau \geqq \sqrt{((a - r)^2 + \{z - (2n + 1)h\}^2)}, \\ &\quad g(y_n) \text{ or } f(y_n) \rightarrow \tau \geqq 2nh,\ h \geqq z \geqq -h,\ \tau \geqq \sqrt{((a - r)^2 + \{z + (2n + 1)h\}^2)}, \end{aligned}\right\} \quad (20)$$

Thus, the stresses in equation (12) are rewritten

$$\left.\begin{aligned}\frac{\tau_{\theta z}}{\tau_0 a} &= \sum_{n=0}^{n_0} (-1)^n \left[\left\{\frac{r}{a^2} H(a-r)H(\tau - x_n) - g(x_n)\right\} + \left\{\frac{r}{a^2} H(a-r)H(\tau - x_n) - g(y_n)\right\}\right],\\ \frac{\tau_{r\theta}}{\tau_0 a} &= -\sum_{n=0}^{n_0} (-1)^n \{f(x_n) - f(y_n)\},\end{aligned}\right\} \quad (21)$$

where n_0 is the maximum integer number of τ/h.

The terms $\tau_0 a\{(r/a^2)H(a-r)H(\tau - x_0) - g(x_0)\}$ and $-\tau_0 af(x_0)$ express respectively $\tau_{\theta z}$ and $\tau_{r\theta}$ in the case of a semi-infinite solid ($z \leqq h$) subjected to torsional impact on the surface. Similarly, $\tau_0 a\{(r/a^2)H(a-r)H(\tau - y_0) - g(y_0)\}$ and $\tau_0 af(y_0)$ correspond to $\tau_{\theta z}$ and $\tau_{r\theta}$ in the case of a semi-infinite solid ($z \geqq -h$) subjected to torsional impact. These stresses propagate in the plate and reflect on the another surfaces. The stresses for $n = 1$ in equation (21) represent the ones generated by these reflected stress waves. Generally, the stresses for n-th term in equation (21) represent the ones generated by the reflected stress waves of $(n-1)$-th term on the surfaces. Figure 2 shows the stress wave fields for $\tau_{\theta z}$.

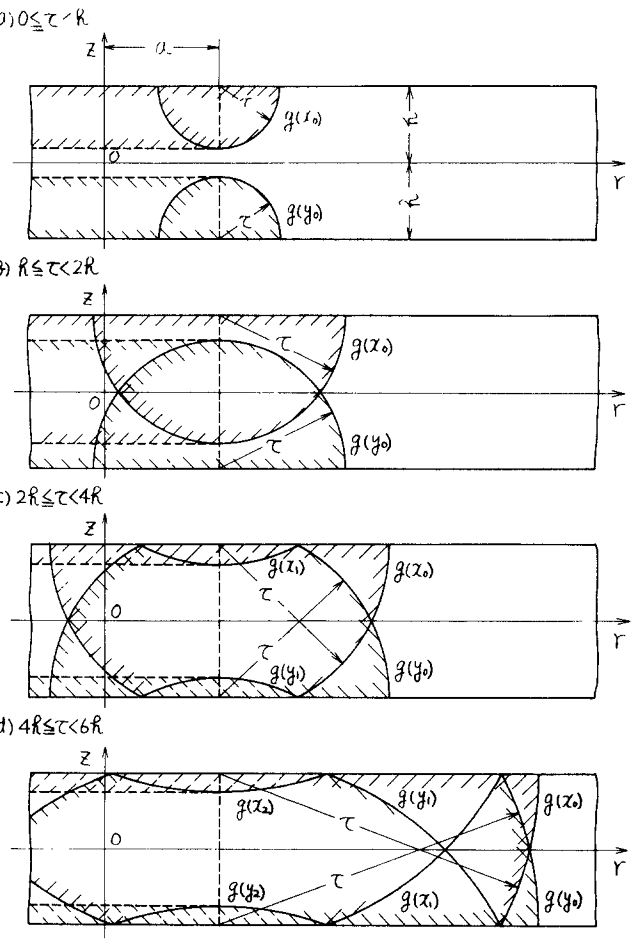

Fig. 2. The stress wave field.

The stresses fluctuate discontinuously at the arrival of the wave fronts of the reflected stress waves. Since the stresses corresponding to each stress wave are calculated independently in closed forms by equation (17), the variations of stresses with time are obtained rigorously by the superposition of the results.

Figure 3 shows $f(x_n)$, ($n = 0, 1, 2, 3, \ldots\ldots$), and $\tau_{r\theta}$ at $r/a = 0{\cdot}9$ on the upper surface for the case of $h/a = 1{\cdot}0$. $f(x_0)$ coincides with $\tau_{r\theta}$ in the case of semi-infinite solid $\tau_{r\theta}$ in the case of the plate is obtained by the superposition of $f(x_n)$, ($n = 0, 1, 2, 3, \ldots\ldots$).

Figure 4 shows $(\tau_{r\theta})_{z=h}$ for $h/a = 1{\cdot}0$. $(\tau_{r\theta})_{z=h}$ is infinite at $r = a$ and becomes small with the increase of distance from this point. The tendency of the fluctuation is similar to than in Fig. 3.

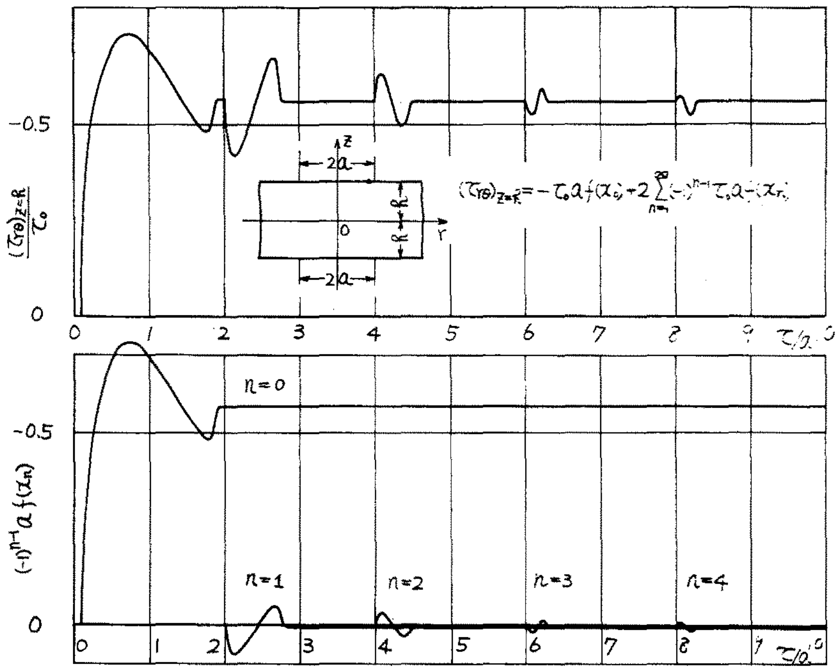

Fig. 3. The interference of the stress waves, ($h/a = 1$, $r/a = 0{\cdot}9$).

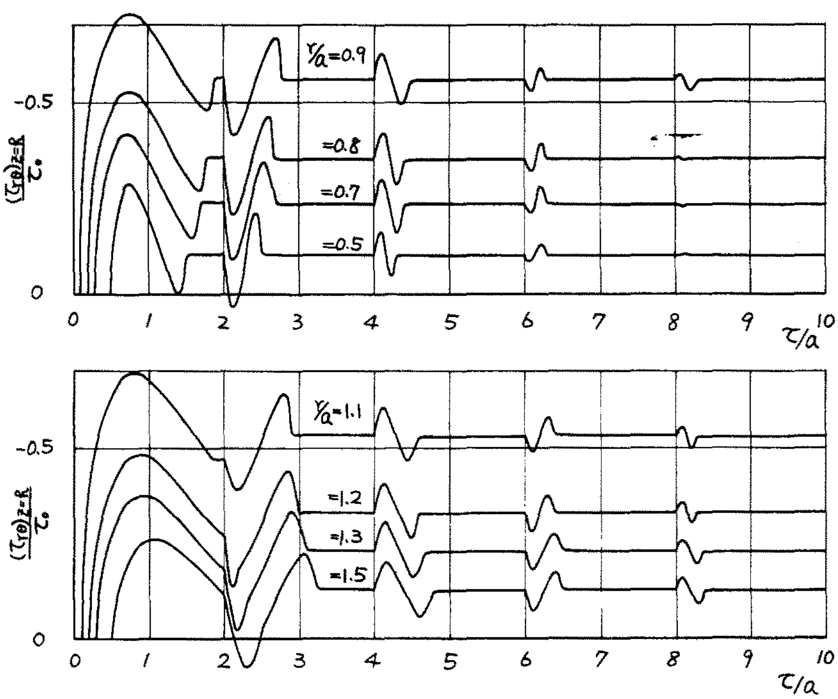

Fig. 4. The variations of $(\tau_{r\theta})_{z=h}$ with time, ($h/a = 1$).

Figure 5 shows $(\tau_{\theta z})_{z=0}$ for $h/a = 1{\cdot}0$. $(\tau_{\theta z})_{z=0}$ in $r \leqq a$ fluctuates discontinuously at $\tau/a = 1, 3, 5, \ldots\ldots$ However, the discontinuous fluctuations tend to vanish with the increase of time. $(\tau_{\theta z})_{z=0}$ in $r > a$ has never such discontinuous fluctuations.

Figure 6 shows $(\tau_{r\theta})_{z=h}$ at $r/a = 0{\cdot}9$ for both cases of $h/a = 0{\cdot}5$ and 2. In the case of the plate thickness being small, $(\tau_{r\theta})_{z=h}$ fluctuates complicatedly because the stress waves repeat numerously the reflection on both surfaces. In the case of the plate thickness being large, the affect of the reflected stress waves is small, and $(\tau_{r\theta})_{z=h}$ becomes nearly equal to the result of the semi-infinite solid.

(a) $r/a = 0{\cdot}5$.

(b) $r/a = 1{\cdot}2$.

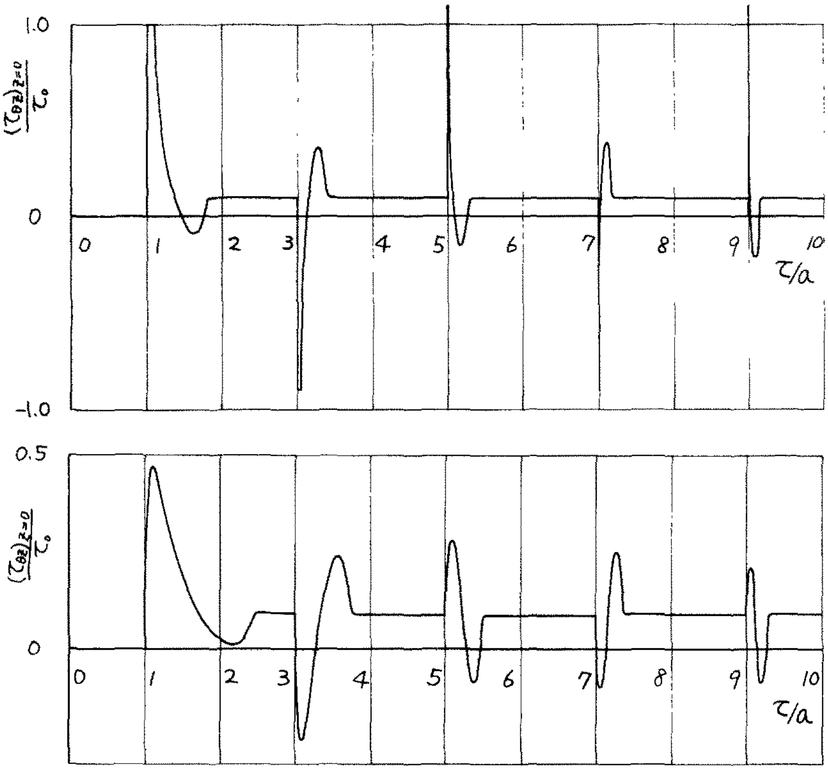

Fig. 5. The variations of $(\tau_{\theta z})_{z=0}$ with time, $(h/a = 1)$.

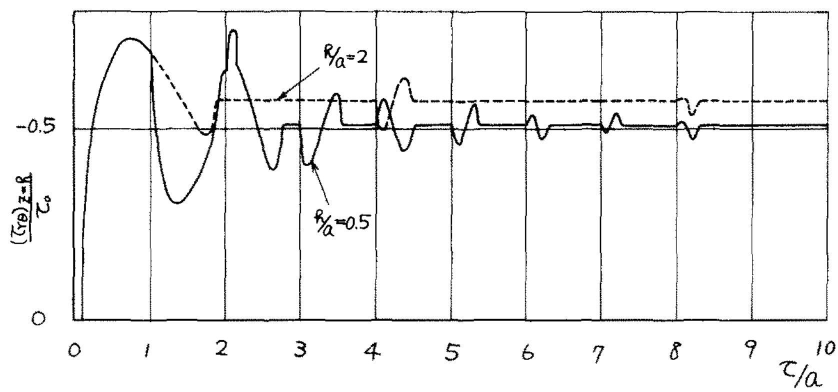

Fig. 6. The relations between $(\tau_{r\theta})_{z=h}$ and thickness, $(r/a = 0{\cdot}9)$.

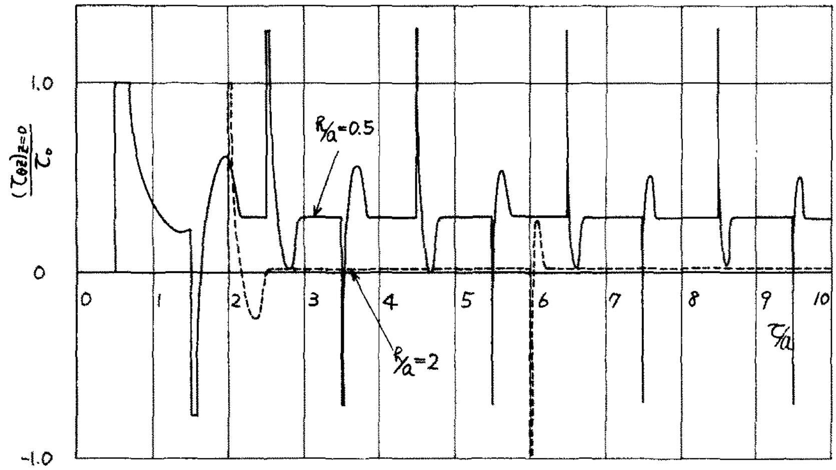

Fig. 7. The relations between $(\tau_{\theta z})_{z=0}$ and thickness, $(r/a = 0{\cdot}5)$.

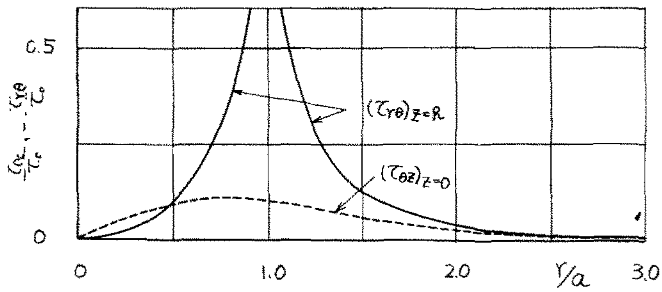

Fig. 8. The asymptotic values of stresses, $(h/a = 1)$.

Figure 7 shows $(\tau_{\theta z})_{z=0}$ at $r/a = 0{\cdot}5$ for both cases of $h/a = 0{\cdot}5$ and 2. The tendency of the fluctuation is similar to the one in Fig. 6.

Figure 8 shows the asymptotic values of stresses for $h/a = 1{\cdot}0$.

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